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A DISTORTION PROBLEM FOR THE SPACE $\ell_\infty^c(\omega_1)$

BY

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ABSTRACT

The Banach space $\ell_\infty^c(\omega_1)$ is the space of bounded ω_1-sequences of countable support. A pointwise-closed subspace $V \leq \ell_\infty^c(\omega_1)$ will be called **unbounded** if $\{\min(\text{supp}(v)) \colon v \in V\}$ is unbounded in ω_1. It is shown that there are Lipshitz functions $f\colon \text{Sph}(\ell_\infty^c(\omega_1)) \to \mathbb{R}$ which have large variation on the unit sphere of any unbounded subspace. This answers a question implicit in Partington [P 80].

1. Introduction

Partington [P 80] has studied equivalent norms on various subspaces of the Banach space $\ell_\infty(\omega_1)$ of bounded real valued functions on ω_1. In particular he was able to prove a strong theorem for the space $\ell_\infty^c(\omega_1)$ of bounded functions of countable support on ω_1 with the supremum norm. A sequence of unit vectors $\langle x_\alpha \colon \alpha < \omega_1\rangle$ in $\ell_\infty^c(\omega_1)$ will be called **successive** if for all $\alpha < \beta < \omega_1$ we have $\sup(\text{supp}(x_\alpha)) < \min(\text{supp}(x_\beta))$.

THEOREM 1 (Partington [P 80]): *Let $|||.|||$ be a norm on $\ell_\infty^c(\omega_1)$ which is equivalent to the supremum norm $||.||$. There is a successive sequence $\langle x_\alpha \colon \alpha < \omega_1\rangle$ of vectors $x_\alpha \in \text{Sph}(\ell_\infty^c(\omega_1))$ and a real number $K > 0$ such that the map $T\colon (\ell_\infty^c(\omega_1), ||.||) \to (\ell_\infty^c(\omega_1), |||.|||)$ which takes the vector $v = (v_\alpha)_{\alpha<\omega_1}$ to $K\sum v_\alpha x_\alpha$ is an isometry.*

It is then natural to ask about the stability of Lipschitz functions on Sph $(\ell_\infty^c(\omega_1))$. We will call a pointwise-closed subspace $V \leq \ell_\infty^c(\omega_1)$ an **unbounded** subspace of $\ell_\infty^c(\omega_1)$ if $\{\min(\text{supp}(v)) \colon v \in V\}$ is unbounded in ω_1. The main purpose of this paper is to construct Lipschitz functions on $\text{Sph}(\ell_\infty^c(\omega_1))$ which have large variation on the unit sphere of any unbounded subspace of $\ell_\infty^c(\omega_1)$.

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THEOREM 2: *There is a Lipschitz function* $f\colon \mathrm{Sph}(\ell_\infty^c(\omega_1)) \to \mathbb{R}$ *such that for all unbounded subspaces* $V \leq \ell_\infty^c(\omega_1)$ *we have*

$$\sup[|f(v) - f(u)|\colon u, v \in \mathrm{Sph}(V)] = 1.$$

In fact we shall construct uncountably many such functions each of which is far from any other.

2. The space $\ell_\infty^c(\omega_1)$

We shall write

$$\ell_\infty(\omega_1) = \{f\colon \omega_1 \to \mathbb{R}\colon \sup(|f(\alpha)|\colon \alpha < \omega_1) < \infty\}.$$

We define

$$\ell_\infty^c(\omega_1) = \{f \in \ell_\infty(\omega_1)\colon \mathrm{card}(\mathrm{supp}(f)) \leq \aleph_0\}.$$

The sup norm on $\ell_\infty^c(\omega_1)$ is written $||.||$ and is easily seen to make $\ell_\infty^c(\omega_1)$ into a Banach space.

3. Unbounded subspaces

Suppose that $\langle x_\alpha\colon \alpha < \omega_1\rangle$ is a successive sequence. Let $\lambda_\alpha \in \mathbb{R}$ for $\alpha < \omega_1$ be a sequence of real numbers. The formal sum $v = \sum_\alpha \lambda_\alpha x_\alpha$ is the function $v\colon \omega_1 \to \mathbb{R}$ such that

$$v(\beta) = \begin{cases} \lambda_\alpha x_\alpha(\beta), & \beta \in \mathrm{supp}(x_\alpha); \\ 0 & \beta \notin \bigcup_\alpha \mathrm{supp}(x_\alpha). \end{cases}$$

We will write $V(\langle x_\alpha\rangle)$ for the set of formal sums $v = \sum_\alpha \lambda_\alpha x_\alpha$ which are in $\ell_\infty^c(\omega_1)$. We now make the following observations:

PROPOSITION 1:

1. *If* $\langle x_\alpha\colon \alpha < \omega_1\rangle$ *is a sequence of successive vectors in* $\ell_\infty^c(\omega_1)$, *then* $V(\langle x_\alpha\rangle)$ *is an unbounded subspace of* $\ell_\infty^c(\omega_1)$.
2. *Let* $V \leq \ell_\infty^c(\omega_1)$ *be an unbounded subspace of* $\ell_\infty^c(\omega_1)$. *There is a successive sequence* $\langle x_\alpha\rangle_{\alpha<\omega_1}$ *such that* $V(\langle x_\alpha\rangle) \subseteq V$.
3. *If* $\langle x_\alpha\colon \alpha < \omega_1\rangle$ *is a successive sequence, then* $V(\langle x_\alpha\rangle)$ *is isometric to* $\ell_\infty^c(\omega_1)$.

4. Stationary sets in ω_1

A set $S \subseteq \omega_1$ is called **stationary** if it meets every closed unbounded subset of ω_1. We write lim for the set of countable limit ordinals.

We shall need the following fact concerning stationary sets.

PROPOSITION 2 (Ulam, see Kunen [K 80]): *There is a family $\{S_\zeta\colon \zeta < \omega_1\}$ of pairwise disjoint stationary sets, $S_\zeta \subseteq \omega_1 \cap \lim$.*

5. Proofs of the results

The proof of Theorem 2 is based on the following lemma. It says that there is a large family of subsets of $\mathrm{Sph}(\ell_\infty^c(\omega_1))$ all of which meet the unit sphere of every unbounded subspace $V \leq \ell_\infty^c(\omega_1)$. If $A, B \subseteq \mathrm{Sph}(\ell_\infty^c(\omega_1))$, we write $d(A,B) = \inf(\|u-v\|\colon u \in A \text{ and } v \in B)$.

LEMMA 1: *There is a sequence $\mathcal{A}_\zeta \subseteq \mathrm{Sph}(\ell_\infty^c(\omega_1))$, $\zeta < \omega_1$ such that:*
1. *For all $\zeta \neq \zeta'$, the distance, $d(\mathcal{A}_\zeta, \mathcal{A}_{\zeta'}) = 1$.*
2. *For all successive sequences $\langle x_\alpha\colon \alpha < \omega_1\rangle$ and for all $\zeta < \omega_1$ we have $\mathcal{A}_\zeta \cap V(\langle x_\alpha\rangle) \neq \emptyset$.*

If $v \in \mathrm{Sph}(\ell_\infty^c(\omega_1))$ is a unit vector in $\ell_\infty^c(\omega_1)$ write $\limsup(v) = 1$ if, for all $\delta < \sup(\mathrm{supp}(v))$, then $\sup(|v(\beta)|\colon \beta > \delta) = 1$.

For each $\zeta < \omega_1$, define

$$\mathcal{A}_\zeta = \{v \in \mathrm{Sph}(\ell_\infty^c(\omega_1))\colon \limsup(v) = 1 \text{ and } \sup(\mathrm{supp}(v)) \in S_\zeta\}.$$

Thus, for instance, if $\alpha \in S_\zeta$, then $1_\alpha = \sum_{\beta<\alpha} \in \mathcal{A}_\zeta$.

Proof of Lemma 1: We will show that the sets $\mathcal{A}_\zeta$ described above satisfy Theorem 1.

We first check that if $\zeta \neq \zeta'$ then $d(\mathcal{A}_\zeta, \mathcal{A}_{\zeta'}) = 1$. To this end, let $u \in \mathcal{A}_\zeta$ and $v \in \mathcal{A}_{\zeta'}$. Let $\alpha = \sup(\mathrm{supp}(u)) \in S_\zeta$ and $\beta = \sup(\mathrm{supp}(v)) \in S_{\zeta'}$.

We may suppose that $\alpha < \beta$, in which case $u \restriction (\alpha,\beta) = 0$. However, $v \in \mathcal{A}_{\zeta'}$ and so $\limsup(v) = 1$. Thus, $\sup(|v(\delta)|\colon \delta > \alpha) = 1$ and so $\|v \restriction (\alpha,\beta)\| = 1$. This means that $d(u,v) \geq 1$ and so $d(\mathcal{A}_\zeta, \mathcal{A}_{\zeta'}) \geq 1$.

Let $\mu \in S_\zeta$ and $\nu \in S_{\zeta'}$. Then $u = 1_\mu \in \mathcal{A}_\zeta$ and $v = 1_\nu \in \mathcal{A}_{\zeta'}$. In addition, $d(u,v) = 1$ and thus $d(\mathcal{A}_\zeta, \mathcal{A}_{\zeta'}) \leq 1$. Combining this and the previous observations we see that $d(\mathcal{A}_\zeta, \mathcal{A}_{\zeta'}) = 1$.

Let $\langle x_\alpha\colon \alpha < \omega_1\rangle$ be successive. Consider the set

$$X = \{\sum_{\alpha<\beta} x_\alpha\colon \beta \in \lim\} \subseteq V(\langle x_\alpha\rangle).$$

We first observe that if $v \in X$, then $\limsup(v) = 1$.

Let $\tau_\beta = \sup(\mathrm{supp}(\sum_{\alpha<\beta} x_\alpha))$. The sequence $\bar{\tau} = (\tau_\beta)_{\beta<\omega_1}$ is thus strictly increasing. To complete the proof of Lemma 1, it is enough to show that the sequence $\bar{\tau}$ is closed in ω_1 (it is clearly unbounded). For then, for all $\zeta < \omega_1$ we will have $S_\zeta \cap \bar{\tau} \neq \emptyset$. Let $\tau_\beta \in \bar{\tau} \cap S_\zeta$. The vector $x = \sum_{\alpha<\beta} x_\alpha$ then satisfies $\limsup(x) = 1$ and $\sup(\mathrm{supp}(x)) \in S_\zeta$ and so $x \in S_\zeta$. To see that $\bar{\tau}$ is closed, let $\alpha_0 < \alpha_1 < \cdots$ be a strictly increasing sequence of countable ordinals. Let $\alpha = \sup_n(\alpha_n)$. Thus $\alpha \in \lim$. Clearly $\tau_\alpha \in \sup(\tau_{\alpha_n})$. Now let $\delta < \alpha$. There is $\beta < \alpha$ such that $\sup(\mathrm{supp}(x_\beta)) \geq \delta$. Because $\beta < \alpha$ there is n such that $\alpha_n \geq \beta$, in which case $\tau_{\alpha_n} = \sup(\mathrm{supp}(x_{\alpha_n})) \geq \delta$. Clearly for every n, τ_α is at least τ_{α_n}. Thus $\tau_\alpha = \sup(\tau_{\alpha_n})$ and this is enough to show that $\bar{\tau}$ is indeed closed. ∎

We may now complete the proof of Theorem 2.

Proof of Theorem 2: Let $\zeta < \omega_1$ be a countable ordinal. Define f_ζ: $\mathrm{Sph}(\ell^c_\infty(\omega_1)) \to \mathbb{R}$ by the equation $f_\zeta(v) = d(\{v\}, \mathcal{A}_\zeta)$. The functions f_ζ are Lipschitz with constant 1.

Let $\zeta \neq \zeta'$ be distinct ordinals in ω_1 and $V \leq \ell^c_\infty(\omega_1)$ be an unbounded subspace. By Proposition 1.2, there is successive sequence $\langle x_\alpha \colon \alpha < \omega_1 \rangle$ such that $V(\langle x_\alpha \rangle) \subseteq V$. By Lemma 1, $\mathcal{A}_\zeta \cap \mathrm{Sph}(V(\langle x_\alpha \rangle)) \neq \emptyset$ and $\mathcal{A}_{\zeta'} \cap \mathrm{Sph}(V(\langle x_\alpha \rangle)) \neq \emptyset$. But, we also know from Lemma 1 that $d(\mathcal{A}_\zeta, \mathcal{A}_{\zeta'}) = 1$. Thus

$$\sup[|f_\zeta(u) - f_\zeta(v)| \colon u, v \in \mathrm{Sph}(V)] \geq 1. \qquad \blacksquare$$

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References

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